

# PLANAR $\text{CAT}(\kappa)$ SUBSPACES

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**ABSTRACT.** Let  $M_\kappa^2$  be the complete, simply connected, Riemannian 2-manifold of constant curvature  $\kappa \leq 0$ . Let  $E$  be a closed, simply connected subspace of  $M_\kappa^2$  with the property that every two points in  $E$  are connected by a rectifiable path in  $E$ . We show that  $E$  is  $\text{CAT}(\kappa)$  under the induced path metric.

## 1. INTRODUCTION

Let  $M_\kappa^2$  be the complete, simply connected, Riemannian 2-manifold of constant curvature  $\kappa \leq 0$ . We show the following.

**Theorem.** *Let  $E$  be a closed, simply connected subspace of  $M_\kappa^2$  with the property that every two points in  $E$  are connected by a rectifiable path in  $E$ . Then  $E$  is  $\text{CAT}(\kappa)$  under the induced path metric.*

See [2] for an alternate treatment where  $\kappa = 0$ , and  $E$  is the set of finite-distance points in the homeomorphic image of a closed disk.

Note the following convention.

**Convention.** We will use the terms *line* and *line segment* to refer to standard geodesic lines and geodesic line segments in  $M_\kappa^2$ . We will use *geodesic* and *geodesic segment* to refer to the geodesics and geodesic segments in  $E$  under the induced path metric.

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## 2. UNIQUE GEODESICS

Let  $E$  be a closed, simply connected subspace of  $M_\kappa^2$  with the property that every pair of points in  $E$  are connected by a rectifiable path in  $E$ . Let  $d$  be the induced subspace metric and  $\bar{d}$  the induced path metric on  $E$ . We will write  $B_d(p, r)$  and  $\bar{B}_d(p, r)$ , respectively, for the open and closed balls of radius  $r$  about  $p \in E$  in the standard metric on  $M_\kappa^2$ .

Since  $E$  is closed in  $M_\kappa^2$ , we know  $(E, d)$  is complete. The proofs of the following two more general results are provided for completeness.

**Lemma 2.1.** *The induced path metric on a complete metric space is complete.*

*Proof.* Let  $(X, d)$  be a complete metric space and  $(X, \bar{d})$  be the induced path metric on  $X$ . Suppose  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence in  $(X, \bar{d})$ . Since  $\bar{d}(x, y) \geq d(x, y)$  for all  $x, y \in X$ , we know  $\{x_n\}$  is also Cauchy in  $(X, d)$ . Hence  $x_n$  converges under  $d$  to some  $x \in X$ . Now a Cauchy sequence converges if and only if it has a convergent

subsequence, so we may assume, by passing to a subsequence if necessary, that  $\bar{d}(x_n, x_m) < 2^{-m}$  for all  $m, n$  with  $n > m$ . So for each  $m$  there exists a path  $c_m: [0, 1] \rightarrow X$  from  $x_m$  to  $x_{m+1}$  with  $l(c_m) \leq 2^{-m}$  by assumption. By linear reparameterization, we have paths  $p_m: [1 - 2^{-m+1}, 1 - 2^{-m}] \rightarrow X$  from  $x_m$  to  $x_{m+1}$  with  $l(p_m) \leq 2^{-m}$ . Pasting these paths together and setting  $p(1) = x$ , we have a continuous map  $p: [0, 1] \rightarrow X$ . Thus  $p$  is a path from  $x_m$  to  $x$  of length at most  $\sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}$ , so  $\bar{d}(x_m, x) \leq 2^{-m+1}$ . Therefore,  $x_m \rightarrow x$  under  $\bar{d}$ .  $\square$

**Corollary 2.2.** *Suppose  $X$  is a complete metric space, and every two points in  $X$  are connected by a rectifiable path. Then the induced path metric on  $X$  is geodesic.*

*Proof.* By definition of path length, every pair of points  $x, y \in X$  has approximate midpoints (see [3, p. 164]). Thus  $X$ , being complete, is geodesic.  $\square$

In Euclidean geometry, the following fact is often useful: Given any line  $L$  and point  $p$ , there is a unique line  $L'$  parallel to  $L$  that passes through the point  $p$ . In hyperbolic geometry, we no longer have a unique parallel line through  $p$ , so we choose a nice one.

**Definition.** Let  $L$  be a line and  $p$  be a point in  $M_{\kappa}^2$ . Let  $K$  be the line segment from  $p$  to the point  $q \in L$  closest to  $p$ . There is a unique line  $L'$  in  $M_{\kappa}^2$  such that the angle between  $L'$  and  $K$  is  $\pi/2$ . We call  $L'$  the *line parallel to  $L$  at  $p$*  and write  $\text{par}(L, p)$  for  $L'$ .

**Lemma 2.3.**  *$(E, \bar{d})$  is uniquely geodesic.*

*Proof.* Suppose  $\sigma: [a, b] \rightarrow E$  and  $\tau: [a, b] \rightarrow E$  are distinct unit-speed geodesics with  $p = \sigma(a) = \tau(a)$  and  $q = \sigma(b) = \tau(b)$ . Note that since both are unit-speed geodesics,  $\sigma(t)$  is in the image of  $\tau$  if and only if  $\sigma(t) = \tau(t)$ , and similarly for  $\tau(t)$ . Since  $\sigma$  and  $\tau$  are distinct, there is some  $t_0 \in (a, b)$  such that  $\sigma(t_0) \neq \tau(t_0)$ , hence  $\sigma(t_0)$  is not in the image of  $\tau$ . Taking the last  $a' \in [a, t_0]$  and the first  $b' \in [t_0, b]$  such that  $p' = \sigma(a')$  and  $q' = \sigma(b')$  are both in the image of  $\tau$ , we have that  $C = \sigma([a', b']) \cup \tau([a', b'])$  is a simple closed curve in  $E$ .

Let  $L$  be the line in  $M_{\kappa}^2$  between  $p'$  and  $q'$ . Let  $R$  be the maximum distance from  $L$  to  $C$ , and let  $t_1$  be the first point of  $[a', b']$  such that either  $d(\sigma(t_1), L) = R$  or  $d(\tau(t_1), L) = R$ . We may assume  $d(\sigma(t_1), L) = R$ . Then, since  $C$  is a simple closed curve and  $a' < t_1 < b'$ , there is some radius  $r > 0$  about  $y = \sigma(t_1)$  such that  $\bar{B}_d(y, r)$  does not intersect  $\tau([a', b'])$ . Let  $A$  be the connected component of  $C \cap \bar{B}_d(y, r)$  containing  $y$ , and let  $s_0 \in [a', t_1]$  and  $s_1 \in [t_1, b']$  satisfy  $\sigma([s_0, s_1]) = A$ .

Now let  $L'$  be the line through  $\sigma(s_0)$  and  $\sigma(s_1)$ . Note that  $d(\sigma(s_0), L) < d(y, L)$  and  $d(\sigma(s_1), L) \leq d(y, L)$  by choice of  $y$ , so  $y \notin L'$  by convexity of  $d$ . By the Jordan curve theorem,  $y$  is the limit of points in the interior region  $D$  bounded by  $C$ . So there is some point  $x \in D$  with  $d(x, y) < d(x, L')$ . Let  $L'' = \text{par}(L', x)$ ; since  $d(x, L') = d(L'', L')$ , we also have  $L' \cap L'' = \emptyset$ . Since  $x$  is in  $D$ ,  $L''$  hits  $C$  on each side of  $x$ ; by construction,  $L''$  first hits  $C$  inside  $\bar{B}_d(y, r)$  in each direction. By choice of  $r$ , we therefore have a straight line segment through  $D$  between two points on  $\sigma([a', b'])$  where  $\sigma$  does not follow the line segment exactly. But  $D \subset E$  since  $(E, d)$  is simply connected, so this contradicts  $\sigma$  being geodesic. Therefore,  $(E, \bar{d})$  is uniquely geodesic.  $\square$

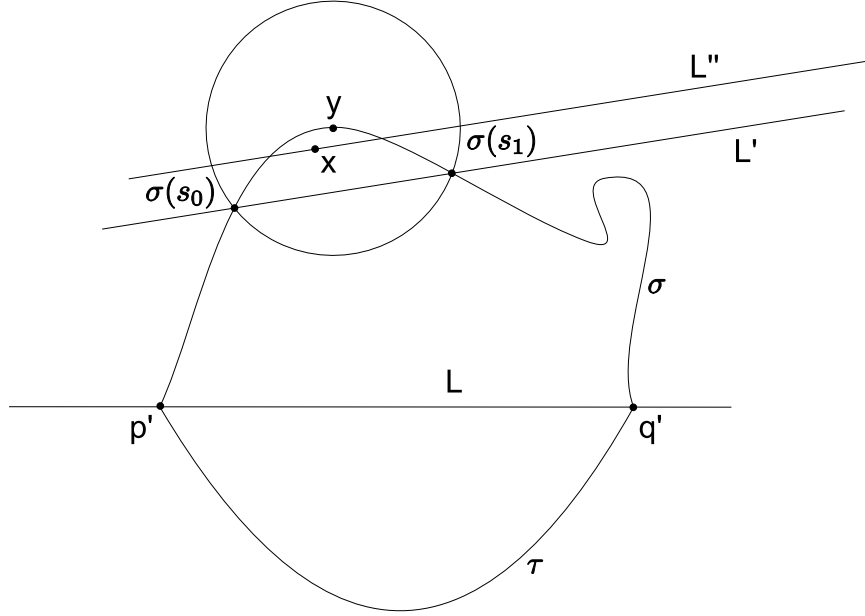


FIGURE 1. Lemma 2.3

## 3. SIMPLE GEODESIC TRIANGLES

We will use the following terminology: Call a geodesic triangle  $T \subset (E, \bar{d})$  *simple* if  $T \subset (E, d)$  is a simple closed curve. For this section, let  $T$  be a simple geodesic triangle in  $(E, \bar{d})$  with interior (under the standard  $M_\kappa^2$  metric)  $S$  and exterior  $U$ .

**Proposition 3.1.** *Let  $L$  be a line in  $M_\kappa^2$  that passes through two distinct points  $p$  and  $q$  that lie on a single edge  $A$  of  $T$ . Let  $L_0$  be the open line segment between  $p$  and  $q$ . If  $L_0$  has empty intersection with  $T$  then  $L_0 \subset U$ .*

*Proof.* Since  $T$  is a simple closed curve in  $(E, d)$  and  $(E, d)$  is simply connected,  $S \subset E$ . Hence if  $L_0$  has empty intersection with  $T$ , we have that  $L_0$  is contained entirely in either  $S$  or  $U$ . But  $L_0 \subset S$  would give us  $L_0 \subset E$ , and this contradicts the hypothesis that  $A$  is the shortest path in  $E$  from  $p$  to  $q$ . Therefore,  $L_0 \subset U$ .  $\square$

**Lemma 3.2.** *Let  $L$  be a line in  $M_\kappa^2$  that passes through the point  $p \in T$ , where  $p$  is not a vertex of  $T$ . Let  $A$  be the edge of  $T$  that contains  $p$ . Suppose that  $r > 0$  is a radius such that  $T \cap B_d(p, r) \subset A$ , and let  $L^-$  and  $L^+$  be the two components of  $L \cap B_d(p, r) \setminus \{p\}$ . Then at least one of  $L^-$  and  $L^+$  has empty intersection with  $U$ . Moreover, if  $L^- \cap T = L^- \cap A \neq \emptyset$  then  $L^+ \cap U = \emptyset$ .*

*Proof.* First suppose, by way of contradiction, that there exist points  $x \in L^- \cap U$  and  $y \in L^+ \cap U$ . Let  $r' > 0$  be some radius with  $r' < r$  such that we have both  $B_d(x, r') \subset U$  and  $B_d(y, r') \subset U$ . Now by the Jordan Curve Theorem,  $T = \partial S$ , so there is some point  $q \in S$  close enough to  $p$  that  $L' = \text{par}(L, q)$  hits points  $x'$  in  $B_d(x, r')$  and  $y'$  in  $B_d(y, r')$ .

Now  $L'$  must be exterior at  $x'$  and  $y'$ , but interior at  $q$ ; furthermore,  $q$  lies between  $x'$  and  $y'$  on  $L'$  by construction. Thus  $L'$  must hit  $T$  somewhere between

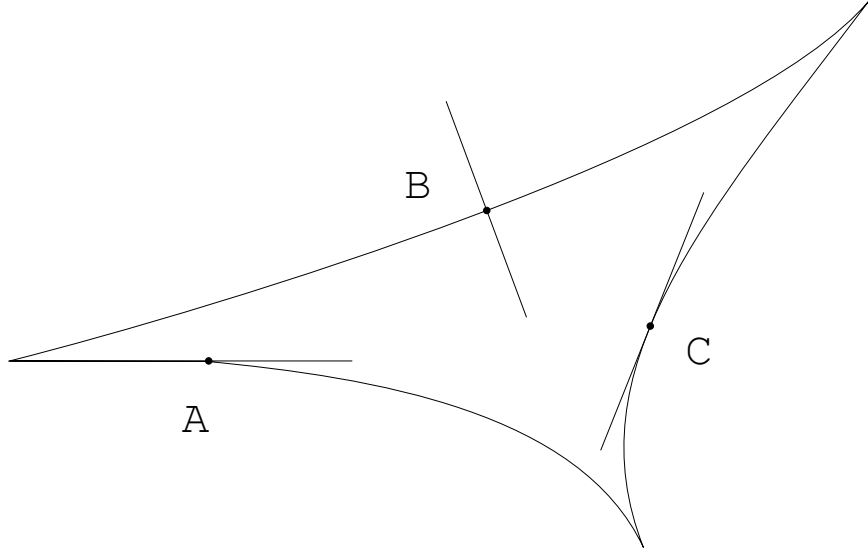


FIGURE 2. Lemma 3.2 allows three types of lines through an edge of a simple triangle: (A) The line intersects the triangle on one side and is locally interior on the other side, (B) the line is locally interior on one side and locally exterior on the other, or (C) the line is locally interior on both sides.

$x'$  and  $q$  and somewhere between  $q$  and  $y'$ . Therefore,  $L'$  hits  $T$  at two points  $x''$  and  $y''$  closest to  $q$  (on opposite sides). By hypothesis on the radius  $r$ , we must have  $x'' \in A$  and  $y'' \in A$ . Hence  $L'$  contains a line segment between two points of  $A$  that is completely interior by construction. This contradicts Proposition 3.1, and therefore at least one of  $L^-$  and  $L^+$  has empty intersection with  $U$ .

Suppose now that there is some point  $z \in T \cap L^-$  and some point  $w \in U \cap L^+$ . Let  $r' > 0$  be some radius with  $r' < r$  such that we have  $B_d(w, r') \subset U$ . The Jordan Curve Theorem guarantees points in  $U$  arbitrarily close to  $z$ , so let  $z' \in U$  be close enough to  $z$  that the line  $L''$  passing through the points  $z'$  and  $p$  enters  $B_d(w, r')$ . But then  $L''$  passes through the point  $p$  and has nonempty intersection with  $U$  on both sides of  $p$ , which contradicts the result of the previous paragraph. Hence  $L^+$  must have empty intersection with  $U$  if  $L^-$  has nonempty intersection with  $T$ .  $\square$

**Corollary 3.3.** *Let  $p_1$ ,  $p_2$ , and  $p_3$  be three distinct points on a single edge  $A$  of  $T$ . Suppose that  $p_1$ ,  $p_2$ , and  $p_3$  lie on a line  $L$  in  $M_\kappa^2$ , with  $p_1$  and  $p_3$  on opposite sides of  $p_2$ . Let  $L_1$  and  $L_2$  be the open line segments from  $p_1$  to  $p_2$  and from  $p_2$  to  $p_3$ , respectively. If  $L_1$  and  $L_2$  both have empty intersection with  $T \setminus A$ , then the arc from  $p_1$  to  $p_3$  along  $T$  follows  $L$ .*

*Proof.* Suppose both  $L_1$  and  $L_2$  have empty intersection with  $T \setminus A$ . Then Proposition 3.1 implies that both  $L_1$  and  $L_2$  must have empty intersection with the interior. Hence Lemma 3.2 gives us that if  $L_1$  has nonempty intersection with  $U$ , then  $L_2$  must follow  $A$ , so  $L_2$  has nonempty intersection with  $T$ , and thus  $L_1$  has empty intersection with  $U$ ; this is a contradiction, so  $L_1$  must have empty intersection

with  $U$ . Thus  $L_1$  follows  $A$  (i.e.,  $L_1 \subset A$ ). Similarly,  $L_2$  must follow  $A$ . Therefore, the arc from  $p_1$  to  $p_3$  along  $T$  follows  $L$ .  $\square$

**Lemma 3.4.** *Suppose the vertices of  $T$  are  $x$ ,  $y$ , and  $z$ . Let  $\triangle'$  be the triangle in  $M_\kappa^2$  with vertices  $x$ ,  $y$ , and  $z$ , and let  $C \subset M_\kappa^2$  be the convex hull of  $\triangle'$ . Then  $T$  is contained in  $C$ .*

*Proof.* Suppose, by way of contradiction, that  $p \in T \setminus C$ . Let  $L$  be the line passing through  $x$  and  $y$ . We may assume that  $p$  lies in the component of  $M_\kappa^2 \setminus L$  that contains no point of  $C$ ; let  $H$  be the closure of this component. Then  $H \cap T$  is compact and nonempty, so it contains at least one point  $p'$  of maximum distance to  $L$ . Let  $L'$  be the line parallel to  $L$  at  $p'$ . Now  $L' \cap T$  is compact and nonempty, so let  $q$  be a point on  $L' \cap T$  of maximum distance to  $p'$ .

Since  $q \notin C$ ,  $q$  is not a vertex of  $T$ . Hence there is a radius  $r > 0$  such that  $B_d(q, r)$  touches no point of any edge of  $T$  other than the one on which  $q$  lies. Let  $L'^-$  and  $L'^+$  be the two components of  $L' \cap B_d(q, r) \setminus \{q\}$ . Lemma 3.2 requires both  $L'^+$  and  $L'^-$  to be in  $T$  since  $L' \cap S$  is empty, but this contradicts our choice of  $q$ . Therefore,  $T \subset C$ , and the theorem is proved.  $\square$

#### 4. LIMIT OUTER ANGLES

If  $p$ ,  $q$ , and  $r$  are distinct point in  $E$ , we will call the angle in  $M_\kappa^2$  at  $p$  between  $q$  and  $r$  the *outer angle at  $p$  between  $q$  and  $r$* , and denote it  $A_p(q, r)$ . Now suppose  $\sigma: [0, 1] \rightarrow E$  and  $\tau: [0, 1] \rightarrow E$  are constant-speed geodesic line segments emanating from the point  $p \in E$ , the images of which intersect only at  $p$ , with  $\sigma(1) = q$  and  $\tau(1) = r$ . By Proposition 3.1 and Lemma 3.4, we have that  $A_p(\sigma(t), \tau(t'))$  decreases monotonically in both  $t$  and  $t'$ , so the *limit outer angle*

$$A'_p(q, r) = \lim_{t, t' \rightarrow 0} A_p(\sigma(t), \tau(t'))$$

is well defined.

The concept of a CAT( $\kappa$ ) space is closely related to the Alexandrov angle at the vertex of a geodesic triangle. Let  $\angle_p^{(\kappa)}(q, r)$  be the angle at  $\bar{p}$  in the comparison triangle  $\triangle(\bar{p}, \bar{q}, \bar{r})$  in  $M_\kappa^2$  for  $\triangle(p, q, r)$ . The Alexandrov angle is defined as

$$\angle_p(q, r) = \lim_{\epsilon \rightarrow 0} \sup_{0 < t, t' < \epsilon} \angle_p^{(0)}(q, r).$$

We will show that the limit outer angle  $A'_p(q, r)$  equals the Alexandrov angle  $\angle_p(q, r)$ . We state the following two results without proof (see [3]). For more discussion on CAT( $\kappa$ ) spaces, we refer the reader to [3] or [1].

**Proposition 4.1.** *For any  $\kappa \in \mathbb{R}$ ,*

$$\angle_p(q, r) = \lim_{\epsilon \rightarrow 0} \sup_{0 < t, t' < \epsilon} \angle_p^{(\kappa)}(q, r).$$

**Proposition 4.2.** *Let  $X$  be a metric space and let  $c$ ,  $c'$  and  $c''$  be three geodesic paths in  $X$  issuing from the same point  $p$ . Then,*

$$\angle(c', c'') \leq \angle(c, c') + \angle(c, c'').$$

As before, let  $T$  be a simple geodesic triangle in  $(E, \bar{d})$  with interior (under the standard  $M_\kappa^2$  metric)  $S$  and exterior  $U$ ; denote the vertices  $p$ ,  $q$ , and  $r$ . Also, let  $\sigma: [0, 1] \rightarrow E$  and  $\tau: [0, 1] \rightarrow E$  be the geodesic line segments from  $p$  to  $q$  and from  $p$  to  $r$ , respectively.

**Lemma 4.3.** *Suppose  $A'_p(q, r) < \frac{\pi}{2}$ , and  $\tau$  follows a line  $L$  in  $M_\kappa^2$  near  $p$  (i.e.,  $\tau([0, \delta]) \subset L$  for some  $\delta > 0$ ). Then there exists  $t_1 > 0$  such that, for any  $t$  with  $0 < t < t_1$ , the line segment from  $\sigma(t)$  to  $L$  perpendicular to  $L$  is contained in  $S \cup T$ .*

*Proof.* Since  $A_p(\sigma(t), \tau(t'))$  decreases monotonically in both  $t$  and  $t'$ , we may find some  $\delta' \in (0, \delta]$  such that  $A_p(\sigma(t), \tau(t')) < \frac{\pi}{2}$  for all  $t$  and  $t'$  with  $0 < t, t' \leq \delta'$ . Let  $D = \overline{B}_d(p, \epsilon)$ , where  $\epsilon > 0$  is small enough that  $D \cap T \subset \sigma([0, \delta']) \cup \tau([0, \delta'])$ . Let  $P$  be projection in  $M_\kappa^2$  onto  $L$ , with domain restricted to the image of  $\sigma$ , and let  $L^+$  be the component of  $L \setminus \{p\}$  that has nonempty intersection with the image of  $\tau$ .

Since  $A'_p(q, r) < \frac{\pi}{2}$ , there is some  $t_0 > 0$  with  $C = \sigma([0, t_0]) \subset D$  such that  $P(\sigma(t)) \in L^+$  for every  $t$  with  $0 < t \leq t_0$ . Since  $P$  is continuous and  $C$  is compact,  $P(C)$  has some point  $q_1 = \sigma(t_1) \in C$  such that  $P(q_1)$  attains the maximum distance from  $p$ . We further require that  $t_1$  be the smallest such value.

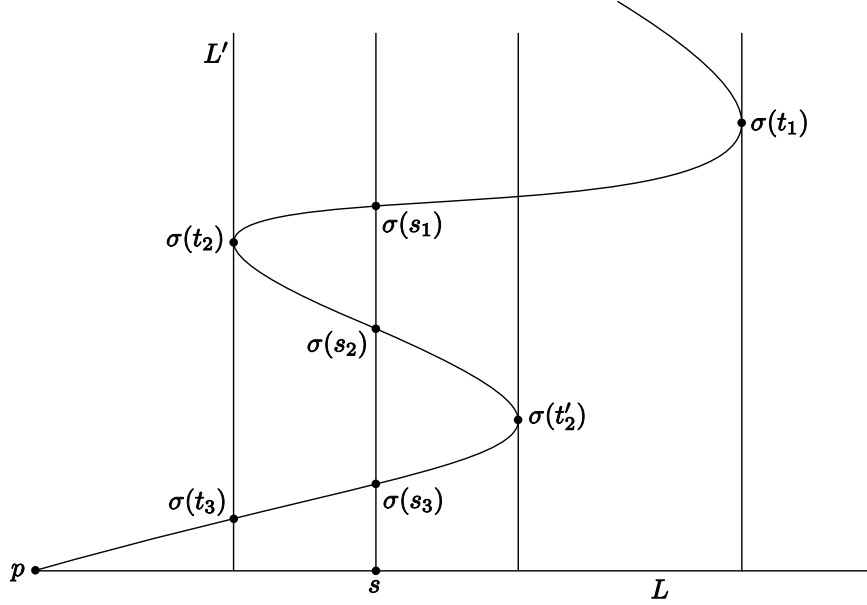


FIGURE 3. Lemma 4.3

Now suppose, by way of contradiction, the line segment  $L'$  from  $q_2 = \sigma(t_2)$  to  $P(q_2)$  contains a point of  $U$  for some  $t_2$  with  $0 < t_2 < t_1$  (note that  $L' \perp L$ ). Let  $t_3$  be the smallest positive value such that  $q_3 = \sigma(t_3)$  lies on  $L'$ . If  $t_3 = t_2$  then the line segment between  $q_2$  and  $P(q_2)$  cuts one of  $S$  or  $U$  into two components; by Lemma 3.4, it must therefore have interior in  $S$ , which contradicts our hypothesis on  $t_2$ . Thus  $0 < t_3 < t_2$ , and  $L'$  has nontrivial intersection with  $U$  between  $q_2$  and  $q_3$ . Hence some  $t'_2$  with  $t_3 < t'_2 < t_2$  must have  $P(\sigma(t'_2))$  farther from  $p$  than  $P(q_2) = P(q_3)$ . Let  $q'_2 = \sigma(t'_2)$ , and let  $s$  be the midpoint between  $P(q_2)$  and  $P(q'_2)$ . By the intermediate value theorem, there must be some  $s_1$  with  $t_2 < s_1 < t_1$  such that  $P(s_1) = s$ . Similarly,  $P^{-1}(s)$  must contain points  $\sigma(s_2)$  and  $\sigma(s_3)$  with  $t'_2 < s_2 < t_2$  and  $t_3 < s_3 < t'_2$ . Thus these three points lie on a line in  $M_\kappa^2$  (orthogonal to  $L$ ), so by Corollary 3.3,  $q_2$  and  $q'_2$  must also lie on this line; this is a contradiction, so no such point  $q_2$  can exist. Therefore, for any  $t$  with  $0 < t < t_1$ , the line segment from  $\sigma(t)$  to  $L$  perpendicular to  $L$  is contained in  $S \cup T$ .  $\square$

**Lemma 4.4.** *Suppose that  $A'_p(q, r) = 0$  and  $\tau$  follows a line  $L$  in  $M_\kappa^2$  near  $p$ . Then  $\angle_p(q, r) = 0$ .*

*Proof.* For simplicity, we assume  $\kappa = 0$  or  $\kappa = -1$ . Let  $\epsilon > 0$  be given. Since  $A_p(\sigma(t), \tau(t'))$  decreases monotonically in both  $t$  and  $t'$ , we may find some  $\delta > 0$  such that  $A_p(\sigma(t), \tau(t')) < \epsilon$  for all  $t$  and  $t'$  with  $\sigma(t), \tau(t') \in \overline{B_d(p, \delta)} \setminus \{p\}$ . Replacing  $\delta$  by a smaller positive constant if necessary, we may assume that every point of  $T$  in  $D$  is in the image of  $\sigma$  or  $\tau$  and that the image of  $\tau$  in  $D$  follows  $L$ . Let  $P$  be the projection from the image of  $\sigma$  onto  $L$ , and let  $t_1$  be the point guaranteed by Lemma 4.3.

Let  $\delta'$  be the distance in  $M_\kappa^2$  from  $p$  to  $P(\sigma(t_1))$ , and note that  $0 < \delta' < \delta$ . Suppose that  $q'$  and  $r'$  are points in  $B_d(p, \delta') \setminus \{p\}$  along the images of  $\sigma$  and  $\tau$ , respectively. Let  $a = d(p, q')$ ,  $b = d(p, r')$ , and  $c = d(q', r')$ , and let  $\phi = A_p(q', r')$ . Also let  $a' = \bar{d}(p, q')$  and  $c' = \bar{d}(q', r')$ ; note that  $a' \geq a$  and  $c' \geq c$ . Since  $\sigma$  is a geodesic, the path straight from  $p$  to  $P(q')$  and then straight to  $q'$ , which stays in  $E$  by choice of  $t_1$ , must have length at least  $a'$ . Hence if  $\kappa = 0$  then

$$a' \leq a(\cos \phi + \sin \phi) \leq a(1 + \sin \phi) \leq a(1 + \sin \epsilon) \leq a(1 + \epsilon),$$

and if  $\kappa = -1$  then by the hyperbolic law of sines,

$$\sinh a' \leq (\cos \phi + \sin \phi) \sinh a \leq (1 + \epsilon) \sinh a.$$

Now suppose that  $c' = c$ . By the law of cosines,

$$\cos \angle_p^{(0)}(q', r') = \frac{(a')^2 + b^2 - c^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2a(1 + \epsilon)b} = \frac{1}{1 + \epsilon} \cos \phi,$$

and by the hyperbolic law of cosines,

$$\cos \angle_p^{(-1)}(q', r') = \frac{\cosh a' \cosh b - \cosh c}{\sinh a' \sinh b} \geq \frac{\cosh a \cosh b - \cosh c}{(1 + \epsilon) \sinh a \sinh b} = \frac{1}{1 + \epsilon} \cos \phi.$$

On the other hand, suppose  $c' > c$ . Note that, by choice of  $t_1$ , the geodesic triangle with vertices  $p$ ,  $\sigma(t_1)$ , and  $P(\sigma(t_1))$  is simple. The interior of this triangle is contained in  $S$ , and  $q' \neq \sigma(t_1)$ . Thus  $L' = \text{par}(L, q')$  must be locally interior on one side of  $q'$ . Let  $L'_0$  be the segment of  $L'$  with  $q'$  as one endpoint, interior in  $S$ , and other endpoint in  $T$ . Let  $p' \in T$  be the other endpoint. Since  $T$  is a simple triangle,  $q' \notin L$ , and therefore  $p' \notin L$ . But  $p' \notin \sigma([0, t_1])$ , so  $p'$  must lie on the line segment from  $\sigma(t_1)$  to  $P(\sigma(t_1))$ . Hence both  $L'_0$  and the line segment from  $q'$  to  $P(q')$  lie in  $S \cup T$ . Thus, if  $P(q')$  lies between  $p$  and  $r'$  on the line  $L$ , then the line segment from  $r'$  to  $q'$  is contained in  $S \cup T$ . Therefore, the outer angle  $A_{r'}(p, q')$  is greater than  $\frac{\pi}{2}$ , and so  $a > c$ .

Now consider the line segment in  $M_\kappa^2$  from  $r'$  to  $q'$ : It hits  $T$  at a first point  $s$  (the edge hit is the one between  $p$  and  $q'$ ). Let  $\gamma$  be the path which travels from  $r'$  to  $s$  along the line segment and then from  $s$  to  $q'$  along  $\sigma$ . Note that the length  $\ell(\gamma)$  of  $\gamma$  is at least  $c'$ . Let  $\alpha$  be the path that travels in a straight line from  $p$  to  $s$  and then straight from  $s$  to  $q'$ , and let  $\alpha'$  be the path that travels in a straight line from  $p$  to  $s$  and then from  $s$  to  $q'$  along  $\sigma$ . Note that  $a \leq \ell(\alpha) \leq \ell(\alpha') \leq a'$ . Hence  $a + c' \leq \ell(\alpha) + \ell(\gamma) = \ell(\alpha') + c \leq a' + c$ , and thus  $a' - c' \geq a - c$ . Therefore,  $a > c$  gives us  $a' - c' > 0$ . Since  $a' \geq a > 0$  and  $c' \geq c > 0$ , we have

$$(a')^{n+1} - (c')^{n+1} = (a' - c') \sum_{k=0}^n (a')^k (c')^{n-k} \geq (a - c) \sum_{k=0}^n a^k c^{n-k} = a^{n+1} - c^{n+1}$$

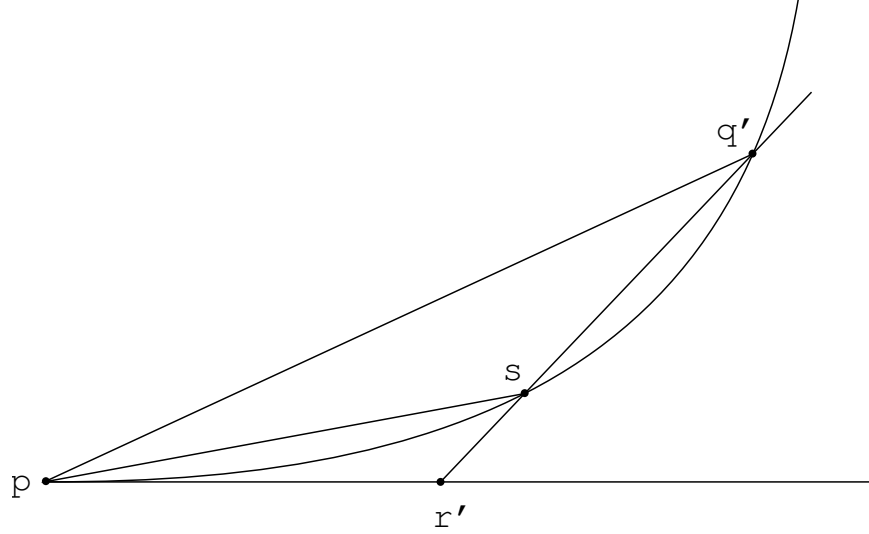


FIGURE 4. Lemma 4.4

for all integers  $n \geq 0$ . Hence for  $\kappa = 0$  we have

$$\cos \angle_p^{(0)}(q', r') = \frac{(a')^2 + b^2 - (c')^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2(a')b} \geq \frac{a^2 + b^2 - c^2}{2a(1 + \epsilon)b} = \frac{1}{1 + \epsilon} \cos \phi,$$

and for  $\kappa = -1$  we have

$$\begin{aligned} \cosh a' - \cosh c' &= \sum_{n=1}^{\infty} \frac{1}{(2n)!} ((a')^{2n} - (c')^{2n}) \\ &\geq \sum_{n=1}^{\infty} \frac{1}{(2n)!} (a^{2n} - c^{2n}) \\ &= \cosh a - \cosh c. \end{aligned}$$

Hence  $\cosh a' - \cosh a \geq \cosh c' - \cosh c$ , so the fact that  $\cosh b \geq 1$  gives us  $\cosh a' \cosh b - \cosh a \cosh b \geq \cosh c' - \cosh c$ , and therefore  $\cosh a' \cosh b - \cosh c' \geq \cosh a \cosh b - \cosh c$ . Thus

$$\cos \angle_p^{(-1)}(q', r') = \frac{\cosh a' \cosh b - \cosh c'}{\sinh a' \sinh b} \geq \frac{\cosh a \cosh b - \cosh c}{(1 + \epsilon) \sinh a \sinh b} = \frac{1}{1 + \epsilon} \cos \phi.$$

Thus, in either case,

$$\cos \angle_p(q', r') \geq \frac{1}{1 + \epsilon} \cos \phi,$$

and therefore we obtain  $\angle_p(q', r') \leq A'_p(q', r') = 0$  as  $\epsilon$  tends to zero. This concludes the proof of the lemma.  $\square$

**Theorem 4.5.** *In a simple geodesic triangle  $T$  with vertices  $p$ ,  $q$ , and  $r$ ,*

$$A'_p(q, r) = \angle_p(q, r).$$

*Proof.* By Proposition 3.1, the rays  $R_{1,t}$  from  $p$  through  $\sigma(t)$  limit monotonically to a ray  $R_1$  as  $t$  tends to zero. Similarly, the rays  $R_{2,t}$  from  $p$  through  $\tau(t)$  limit monotonically to a ray  $R_2$  as  $t$  tends to zero.



Suppose first that  $R_1 \neq R_2$ . By construction,  $R_1$  and  $R_2$  are locally contained in  $S \cup T$  near  $p$ . Let  $s_1$  be the last point of  $R_1$  contained in  $S \cup T$ . Clearly,  $s_1 \in T$ ; if  $s_1$  lies along  $\sigma$  then  $s_1$  must equal  $q$  by Lemma 3.2. Since  $R_1$  is locally contained in  $S \cup T$  near  $p$ , we have  $s_1 \neq p$ , and thus  $s_1$  cannot lie along  $\tau$ . Therefore,  $s_1$  lies along the geodesic arc between  $q$  and  $r$ . Similarly, the last point  $s_2$  of  $R_2$  that is contained in  $S \cup T$  must lie along the geodesic arc between  $q$  and  $r$ . Note that  $\angle_p(s_1, s_2) = A'_p(q, r)$ , since both measure the angle between  $R_1$  and  $R_2$ .

If  $\sigma$  follows  $R_1$  for some positive distance beyond  $p$ , then  $\angle_p(q, s_1) = 0$  by definition. On the other hand, if  $\sigma$  does not follow  $R_1$  for any positive distance beyond  $p$ , then the geodesic triangle  $T_1 = \triangle(p, q, s_1)$  is simple, and  $\angle_p(q, s_1) = 0$  by Lemma 4.4. Thus in either case,  $\angle_p(q, s_1) = 0$ ; similarly,  $\angle_p(s_2, r) = 0$ . Hence

$$\angle_p(q, r) \leq \angle_p(q, s_1) + \angle_p(s_1, s_2) + \angle_p(s_2, r) = \angle_p(s_1, s_2)$$

and

$$\angle_p(s_1, s_2) \leq \angle_p(s_1, q) + \angle_p(q, r) + \angle_p(r, s_2) = \angle_p(q, r)$$

by Proposition 4.2. Therefore  $\angle_p(q, r) = \angle_p(s_1, s_2) = A'_p(q, r)$ .

Finally, suppose  $R_1 = R_2$ ; note that this gives  $A'_p(q, r) = 0$ . If  $\sigma$  follows  $R_1$  for some positive distance beyond  $p$ , then  $\angle_p(q, r) = A'_p(q, r) = 0$  by Lemma 4.4. Thus we may assume, by symmetry, that neither  $\sigma$  nor  $\tau$  follows  $R_1$  for any positive distance beyond  $p$ . Then by construction of  $R_1 = R_2$ , the last point  $s$  of  $R_1$  contained in  $S \cup T$  must be along the geodesic arc from  $q$  to  $r$ . Hence the geodesic triangles  $T_1 = \triangle(p, q, s)$  and  $T_2 = \triangle(p, s, r)$  are simple, and since  $A'_p(q, s) = A'_p(s, r) = 0$  by construction,  $\angle_p(q, s) = \angle_p(s, r) = 0$  by Lemma 4.4. Therefore,

$$\angle_p(q, r) \leq \angle_p(q, s) + \angle_p(s, r) = 0,$$

and the theorem is proved.  $\square$

**Theorem 4.6.**  $(E, \bar{d})$  is a complete  $CAT(\kappa)$  space.

*Proof.* Note that every geodesic triangle with distinct vertices either has 0 angle at all 3 vertices, or it can be trimmed to a simple triangle. Moreover, this trimming does not decrease the angles at the vertices. So let  $p, q$ , and  $r$  be the vertices of a simple triangle  $T$ . As in the proof of Theorem 4.5, we have two (possibly equal) limit rays  $R_1$  and  $R_2$  from  $p$ . Cutting along these rays gives three (possibly degenerate) triangles. The middle triangle has Alexandrov angle at  $p$  at most the limit outer angle, since only the edge opposite  $p$  can be longer than the distance in  $M_\kappa^2$ . The two outside triangles have Alexandrov angle 0 at  $p$  by Lemma 4.4. So by Alexandrov's Lemma ([3, p. 25]),  $\angle_p(q, r) \leq \angle_p^{(\kappa)}(q, r)$ . Therefore,  $(E, \bar{d})$  is  $CAT(\kappa)$ .  $\square$

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